1. \( X_1, \ldots, X_n \) are iid \( N(\mu, \sigma^2) \) and \( \sigma \) is known.

(a) The joint density of the \( X \)'s is \( f(x_1, \ldots, x_n | \mu) = \)

(b) Derive the maximum likelihood estimator \( \hat{\mu} \) of \( \mu \).

(c) The likelihood ratio for the hypothesis

(*) \( H_0 : \mu = 0 \) against \( H_A : \mu \neq 0 \)

is \( \lambda(x) = \)

(d) Prove that

\[
\sum_{j=1}^{n} (x_j - t)^2 = \sum_{j=1}^{n} (x_j - \bar{x})^2 + n(\bar{x} - t)^2
\]

and use this to prove that

\( \lambda(x) \leq c \) if and only if \( |\bar{x}| \) is too large.
(e) Show that the likelihood ratio test of (*) of size $\alpha$ rejects $H_0$ if $\sqrt{n} \left| \frac{\bar{x}}{\sigma} \right| > z_{\alpha/2}$.

2. Assuming the data below constitutes a random sample from a normal distribution, does the data present significant evidence at $\alpha = .05$ that the mean of the population differs from 7? You may use, without derivation, the likelihood ratio test we developed in class for $H_0 : \mu = \mu_0$ against $H_A : \mu \neq \mu_0$. Be sure to state the test before you use it.

   1.92 3.84 5.50 2.62

3. If $x_1, \ldots, x_n$ are fixed numbers, not all zero, and if $Y_1, \ldots, Y_n$ are independent,

   $Y_i \sim N(\theta x_i, 1)$

   then

   a) the joint density of the $Y$'s is

   $f(y_1, \ldots, y_n | \theta) =$
b) Give the form of the Neyman-Pearson most powerful test of
\[ H_0 : \theta = 0 \quad \text{against} \quad H_A : \theta = 1 \]

c) What is the probability distribution of \( a_1 Y_1 + ... + a_n Y_n \)?

d) Give the Neyman-Pearson most powerful test of the hypotheses in (b) of size \( \alpha = 0.05 \).

e) Is the test in d) UMP (uniformly most powerful) of size \( \alpha = 0.05 \) for testing
\[ H_0 : \theta = 0 \quad \text{against} \quad H_A : \theta > 0 ? \]

Why?