You may use your calculator, your crib sheet, and the tables supplied. Please use no other outside source of information. **Show your work.**

1. Suppose $X_1, \ldots, X_n$ are, for $\theta$ fixed, i. i. d. with probability density

$$f(x \mid \theta) = \begin{cases} \theta e^{-\theta x} & \text{if } x > 0 \\
0 & \text{otherwise} \end{cases}.$$ 

1a) Give the joint probability density of the $X$'s given $\theta$.

1b) Give the Bayes estimator of $\theta$ under squared error loss if the prior on $\theta$ is the gamma distribution $G(\alpha, \beta)$.

2. Consider the random variable $\Theta$ whose probability density is

$$\pi(\theta) = \begin{cases} \theta^{\alpha-1} e^{-\theta/\beta} / \Gamma(\alpha) \beta^{\alpha} & \text{if } \theta > 0 \\
0 & \text{otherwise} \end{cases}.$$ 

2a) Find the mode (location of the maximum) of this density if $\alpha \geq 1$.

2b) Show that $E ( \Theta^z ) = \frac{\Gamma(\alpha+z)}{\Gamma(\alpha)} \beta^z$ if $z$ is any real number such that $z + \alpha > 0$. (Hint: For every $u > 0$ and $v > 0$ the function of $\theta$ given by $\pi(\theta) = \begin{cases} \theta^{u-1} e^{-\theta/v} / \Gamma(u) v^u & \text{if } \theta > 0 \\
0 & \text{otherwise} \end{cases}$ is a probability density.)

3. If $X_1, \ldots, X_n$ are i. i. d. with probability density

$$f(x \mid \theta) = \begin{cases} \theta e^{-\theta x} & \text{if } x > 0 \\
0 & \text{otherwise} \end{cases}.$$ 

3a) Find the maximum likelihood estimator $\hat{\theta}$ of $\theta$.

3b) What is the probability distribution of $\sum_{i=1}^{n} X_i$?

3c) Is $\hat{\theta}$ an unbiased estimator of $\theta$? (Hint: Use the result of part b of problem 2) If it is not, give an unbiased estimator $\tilde{\theta}$ of $\theta$.

3d) State a relationship between maximum likelihood estimators and efficient estimators.

3e) Is $\tilde{\theta}$ efficient?
4. Poisoning by DDT causes tremors and convulsions. In a study of DDT poisoning, four rats randomly selected from a certain strain were fed a measured amount of DDT and their absolute refractory periods were measured (the absolute refractory period is the time required for a nerve to recover after a stimulus). Assuming the absolute refractory period is normally distributed, give a 90% confidence interval for the mean absolute refractory period for the rats of the strain from which the following four measurements were taken.

   1.25  1.43  1.35  1.52

5. An entomologist samples a field for egg masses of a harmful insect by placing a yard-square frame at random locations and carefully examining the ground within the frame. A random sample of 50 locations selected from a county’s pasture land found egg masses in 13 locations. Give a 95% confidence interval for the proportion of all possible locations that are infested.