1. Let $X_1, X_2, X_3$ be a random sample from a distribution of the continuous type having probability density function $f(x) = 2x I_{(0,1)}(x)$. Compute the probability that the smallest of these $X_i$ exceeds the median of the distribution.

2. Let $Y_1 < \ldots < Y_n$ be the order statistics from a random sample of size $n$ from an exponential distribution with mean 1.
   (a) Show that $Z_1 = nY_1, Z_2 = (n-1)(Y_2 - Y_1), \ldots, Z_n = Y_n - Y_{n-1}$ are stochastically independent and that each has an exponential distribution.
   (b) Prove that all linear functions $\sum_{i=1}^{n} a_i Y_i$ of $Y_1, \ldots, Y_n$ can be expressed as linear combinations of stochastically independent random variables.

3. Let $F$ be a continuous distribution and $X_{(1)} < \ldots < X_{(n)}$ be the order statistics from a random sample of size $n$. Set $C_1 = F(X_{(1)}), C_2 = F(X_{(2)}) - F(X_{(1)}), \ldots, C_{n+1} = 1 - F(X_{(n)})$. Show that the joint probability density function of $C = (C_1, C_3, C_4, \ldots, C_{n+1})$ is

$$h(c_1, c_3, c_4, \ldots, c_{n+1}) = \begin{cases} n! & \text{if } c \in D \\ 0 & \text{otherwise} \end{cases},$$

where

$$D = \{ c = (c_1, c_3, c_4, \ldots, c_{n+1}) : c_i \geq 0 \text{ for } i = 1, 3, 4, \ldots, n+1 \text{ and } \sum_{i=1}^{n+1} c_i \leq 1 \}. $$
4. Prove the following regarding $S_n^2 = \frac{1}{n-1} \sum_{j=1}^{n} (X_j - \bar{X})^2$, where $X_1, \ldots, X_n$ are iid from a probability distribution for which $E[|X|^4] < \infty$. (Hint: You'll need Chebyshev's inequality, the weak law of large numbers, and the fact that if $L \rightarrow U$ and $p \rightarrow 0$, then $L \rightarrow U$.)

(a) $\frac{S_n}{\sigma} \rightarrow 1$.

(b) $\sqrt{\frac{n-1}{2}} \left( \frac{S_n^2}{\sigma^2} - 1 \right) \rightarrow N(0, 1 + \frac{\gamma_2}{2})$. Here $\gamma_2 = E\left[ \frac{(X-\mu)}{\sigma}^4 \right] - 3$ is the kurtosis of the distribution of $X$.

5. Let $X$ have the Beta distribution with parameters $\alpha > 0$ and $\beta > 0$, written $X \sim \text{Be}(\alpha, \beta)$. The probability density of $X$ is

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} I_{(0,1)}(x).$$

(a) Find the mean and variance of $X$.

(b) Find $E(X^k)$ for $k \in \{1, 2, \ldots\}$ if $X \sim \text{Be}(m,n)$ and $m$ and $n$ are positive integers.

6. Find an approximate 90% distribution-free confidence interval for $\pi_4$ for the continuous distribution from which the following data constitute a random sample.

$100.89 \ 98.76 \ 109.75 \ 97.94 \ 110.04 \ 100.95 \ 119.27 \ 98.97$

$100.74 \ 100.51 \ 99.11 \ 101.46 \ 106.02 \ 101.02 \ 98.35 \ 102.97$

7. Find an approximate 95% upper 75% tolerance region for the continuous distribution from which the following data constitute a random sample.

$71.86 \ 102.79 \ 68.55 \ 131.42 \ 95.68 \ 80.50 \ 89.27$
8. Let \( y_1, \ldots, y_N \) denote \( N \) arbitrary numbers. Let \( Y \) denote the random variable which is the result of sampling one of the \( y \)-values at random (each \( y_i \) has probability \( 1/N \) of being sampled.)

a) Find \( E(Y) \).

b) Find \( \text{Var}(Y) \).

c) Specialize these to the case \( y_i = i, i = 1, 2, \ldots, N \) and show that
\[
E(Y) = \frac{N+1}{2}
\]
and
\[
\text{Var}(Y) = \frac{N^2 - 1}{12}.
\]

(Hint: Define the indicator random variables
\[
Z_i = \begin{cases} 
1 & \text{if } y_i \text{ is sampled} \\
0 & \text{otherwise}
\end{cases}
\]
and write \( Y = \sum_{i=1}^{N} Z_i \cdot y_i \ architectural notation.)