Let $V$ and $W$ be finite dimensional vector spaces over the reals. Let $A = \{v_1, \ldots, v_m\}$ be a basis for $V$ and $B = \{w_1, \ldots, w_n\}$ be a basis for $W$. Then each vector $v \in V$ can be represented uniquely as a linear combination
\[
v = \sum_{j=1}^{m} a_j v_j,\]
where the $a$'s are real numbers, called the coordinates of $v$ in the basis $A$. The coordinate vector of $v$ relative to the basis $A$ is $[v]_A$ so that
\[
\begin{bmatrix}
a_1 \\
a_2 \\ \\
\vdots \\
a_m
\end{bmatrix} = [v]_A.
\]

To know a vector it is clearly enough to know its coordinates in any basis.

**Lemma.**

If $T$ is a linear transformation $T : V \to W$, then there is a matrix $M$ such that for all vectors $v \in V$ and $w \in W$ one has
\[
[Tv]_B = M [v]_A
\]
where the right hand side is the usual multiplication of a vector $(m \times 1)$ by a matrix $(n \times m)$. We write
\[
M = [T]_B^A
\]
and call $M$ the matrix of $T$ relative to the bases $A$ and $B$.

**Proof:**

To see that this is correct note that if $v = \sum_{j=1}^{m} a_j v_j$ then by linearity
\[
Tv = \sum_{j=1}^{m} a_j T v_j.
\]
Furthermore, since $T v_j$ is in $W$ there are scalars $b_{j1}, \ldots, b_{jn}$ such that for each $j$
\[
T v_j = \sum_{i=1}^{n} b_{ji} w_i.
\]
Therefore
\[
Tv = \sum_{j=1}^{m} a_j \sum_{i=1}^{n} b_{ji} w_i = \sum_{j=1}^{m} \sum_{i=1}^{n} a_j b_{ji} w_i
\]
and it follows that the coordinates of $Tv$ in the basis $B$ are, for $i = 1, \ldots, n$,
\[
\sum_{j=1}^{m} a_j b_{ji}.
\]
Thus $[Tv]_B = B' a = B' [v]_A$ and we can take $M = B'$. ☐